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On the dynamics of illicit drug consumption in a given population

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We present and analyse a model that describes the illicit drug usage dynamics in a population consisting of both drug users and non-users. Three categories of drug users are considered, namely, the experimental category, the recreational category and the addict category. In this model, consumption of drugs such as Cannabis is considered and the model describes the dynamics of the non (N), experimental (E), recreational (R) and addict (A) user categories, respectively, within a given population. We term it as the NERA model (N: non-users, E: experimental, R: recreational users, A: addicts). We calibrated our model using the data from Korf (2001). That is, we classified the cannabis consumers from the population of 12+ in Amsterdam, for the period 1987–1997, into the non-user, experimental, recreational and addict category, respectively, for the verification and validation of our dynamical system. We then discuss and analyse the stability of the critical points when drug consumption is (i) legal and (ii) illegal in the society. The results of our numerical experiments indicate that the solution of our model agrees quite accurately with the existing data. Our analysis also reveals the existence of a drug-free equilibrium under certain conditions. The model can eventually be used as a policy control mechanism in tackling the consumption of illicit drugs in a given society.

Keywords: non-users; experimental users; recreational users; addicts; dynamical system; legal; illegal; critical points; stability; bifurcation; Lyapunov function.

1. Introduction

The term ‘drug’ is defined as any substance that can be used to modify a chemical process or processes in the body, such as treating an illness, relieve a symptom, enhance a performance or ability or alter states of mind (http://www.maltatoday.com.mt/2006/04/02/t17.html).

There exists safe drugs as well as illicit drugs. The former are mainly medicinal drugs where some are prescribed by doctors to cure patients by relieving them from pain, while others are non-prescription drugs, such as aspirin and paracetamol (http://www.watchover.org/library/g/2001/7/8/article_01.htm). Abuse of medicinal drugs can lead to serious health problem (http://www.watchover.org/library/g/2001/7/8/article_01.htm).

Illicit drugs are those drugs, whose possession, use or sale are illegal (Glossary of Alcohol and Drug and Other Terms, 2003). Clearly drugs such as tobacco and alcohol do not fall in the above two categories because their possession, use and sale are permitted by the law while they are harmful to health.

In fact if we look from a scientist point of view, we will see that there is no direct correlation between the legal status of a drug and its dangers (Drug risks, 2004). Some of the most harmful drugs are legal, while some of the least dangerous one are strictly prohibited; this is because drugs get banned mainly for historical, social and racial reasons, instead of medical and scientific reasons (Drug risks, 2004).

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Illicit drug consumption is a world-wide problem as we live in a world filled with temptation. Drugs are being utilized as artificial problem solvers. While the lure of drug is that they can make a problem disappear for a short period of time, they are by no means the answer for solving the problem (Faila, 1999).

Anyone could be a 'prey' to illicit drugs. Drug users could be of any gender, of any age, having attained any level of education, from any profession, of any religion or race.

Drugs Users are generally classified into three categories, depending on their frequency consumption and the control they have over the drug (http://www.drugscope.org.uk/wip/24/parents.htm, 2005):

(1) Experimental (http://www.drugscope.org.uk/wip/24/parents.htm; Level of Drug Use; http://www.wbebp.co.uk/support/unit2_drug_support_centre_ii.htm)

These users are those who are trying drugs for the first few times, that is they explore drugs or use it over short period of time. Most of them are ill informed, lacking knowledge and prior thought about drug use. Their choice of drugs vary, depending on the availability, sub-culture, fashion and peer group influences.

(2) Recreational (http://www.drugscope.org.uk/wip/24/parents.htm; Level of Drug Use; http://www.wbebp.co.uk/support/unit2_drug_support_centre_ii.htm)

They use drugs in a regular but fairly controlled way, that is, they have established control over what, where, when and how much should be used. They are usually particular in their choice of drugs and most of the time take them in social groups.

(3) Drug addicts (http://www.drugscope.org.uk/wip/24/parents.htm; Level of Drug Use; http://www.wbebp.co.uk/support/unit2_drug_support_centre_ii.htm)

These individuals take drugs frequently and continuously feel the need to use them for physical or psychological reasons. They feel that they have to be on drugs almost all the time to face the world. It is very much an escape from normal life and very different from reaching out into a new and exciting lifestyle that experimenters and recreational users are looking for. Also, dependant users are indiscriminating in substances because obtaining the drug is more important than its quality.

Most drugs users have experienced drugs during their youth, as it is a time of experimentation and taking risks in the process of discovering individual identities. In fact, research has highlighted that drug use among young people is often experimental and/or recreational. Most young people experiment with using drugs to cure their curiosity about the sensations they produce. The danger with these experimentations is not knowing when to stop. The majority of them will eventually take drugs to feel the euphoria that drugs often deliver. Some will take drugs to feel less inhibited, to be part of a group, for fun or simply to overcome boredom. However, according to research, the most common reason why people take drugs is to change the way they feel, that is to conceal their feelings of pain (Faila, 1999).

Though everyone reacts differently to drugs, noo-ne is safe from them, if misused. As seen above, drugs in general affect people physically, psychologically or both, especially those using them on a regular basis. Drug abusers are in danger of inflicting damage to their physical and mental health as well as their social well being. A well-known fact is that drug abuse can put a person at risk of getting sexually transmitted diseases such as HIV, this is because, though people associate these diseases with injection

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drug use only (sharing needles and syringes), drug intoxication affects judgement and can lead to unsafe sexual practices which put people at risk of getting HIV or transmitting it to someone else (The Science Behind Drug Abuse, 2005). Also, drug abuse and addiction can affect a person's overall health, thereby altering susceptibility to HIV and the progression of AIDS (The Science Behind Drug Abuse, 2005). In addition, there are some cases of deaths that have been reported to be directly attributable to illicit drug overdose (Faila, 1999). Moreover, not only are drug users at risk of being caught in possession of drugs, but they also represent a danger to society as they often resort to crime to fund their purchases of illicit drugs. To be more explicit, they are prone to commit crimes such as larceny, burglary, shoplifting and prostitution to finance their habit. Another common view is that drugs change the nervous system in many ways and therefore cause drug users to be violent (Jofre-Bonet & Sindelar, 2001).

In many countries, including Mauritius, debates are being held on whether to legalize some illicit drugs, for example Cannabis (http://www.legalisedrugs.co.uk/, 2002; Deglamorising Cannabis 1995; Gandia: Debat enclenche 2006; Un collectif refuse la legalisation du "gandia" 2006). If drugs are legalized, they will be accessible to almost everyone, as cigarettes and alcohol presently are. Therefore, many people, who had been strictly against drugs in the past, might change their point of view about these products. Probably some of them would like to try drugs, seeing no harm in them all of a sudden. Moreover, they might start interacting with users from the recreational category, whom they would not have approached before, thinking that these were not respectable individuals (committing offences by consuming illicit things). It is clear that being in the company of these recreational users will have a great influence on them trying drugs. In this model, we take into consideration that, drugs being legalized, the influence that individuals exert on each other will be very high. More precisely, the individuals from the recreational category will influence individuals from the experimental category, while the non-users will be influenced by both.

In this work, we propose to assess and quantify the effect of legalizing drugs in a given population. We analyse the variation of the size of each category of drug user with respect to the mutual influence that each of them exert on the others.

2. Organisation of paper

The paper is organized as follows. In the next section we analyse different existing models and the attempts at modelling in the field of drug consumption and prevention. We then give a schematic representation of the NERA model. Then we enumerate the assumptions and set up the dynamical system governing the model. The resulting system of differential equations are analysed initially when drug consumption is legal and then when it is illegal. We conduct numerical experiments to verify and validate the model with respect to data collected from Korf (2001).

3. Existing models

Among recent models of illegal drug consumption in a given population, one can find the work of Grass *et al.* (2008). In that work, descriptive models of drug demand and optimal control model of drug epidemics are given. In Caulkins & Reuter (2004), the authors propose some policy recommendations. Bultman *et al.* (2009) studied a time-discrete stochastic reformulation of an optimal control model of illicit drug consumption in which the retail drug price is influenced by exogenous random forces.

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The one-dimensional drug model is possibly the simplest of all drug-prevalence models. It consists of only one state variable, which is the total number of drug users at time t. It also has two control variables: expenditure for treatment at time t, $u(t)$, and law enforcement measures, $v(t)$ (Grass *et al.*, 2008).

The LH model depicts two groups of users, heavy users, denoted by H and light users, denoted by L and the group of non-users. The number of non-users in this model is assumed to be very large and constant so that they do not need to be modelled explicitly. This model of illicit drug consumption maps out the initiation of non-users to become light users, the rate of escalation of light users towards the heavy users category and the rate of desistance of heavy users. Furthermore, the model also assumes the movement of heavy users towards the light users category (Behrens & Tragler, 2000; Grass *et al.*, 2008). From this LH Model, other models have been derived such as the initiation model (Behrens & Tragler, 2000; Grass *et al.*, 2008) and the model of US Cocaine Consumption (Grass *et al.*, 2008).

The model of controlled drug demand is an extension of the LH Model, whereby two more controls are added representing spending on drug prevention and spending on drug treatment. This model sheds light on the fact that both prevention and treatment are helpful in suppressing drug consumption but not necessarily when applied at the same time (Grass *et al.*, 2008).

4. The NERA model (N: non-user, E: experimental users, R: recreational users, A: addicts)

We consider the mutual influence that drug users can have on non-users and on each other, thereby introducing non-linear terms in the model. Figure 1 gives a schematic representation of drug consumption in a given population.

The different arrows, in fact, represent the movement from one category to another. The schematic representation is realistic in the sense that non-users make up the larger part of the population, while addicts make up the smaller portion of the population. In our models, however, we ignore the fact that experimental users can become addicts directly because, to be more realistic, there are not many individuals who become dependent on drugs at an early stage of drug taking (http://www.needle.co.nz/fastpage/fpengine.php/templateid/30). Furthermore, we consider movement into and out of the population P. This movement is a result of individuals becoming too old to still form part of the population (thus moving out) or having become mature enough and been introduced into the population (thus moving in). We neglect movement out of the population due to death, because death rate is negligible compared to the aging rate in this population. Individuals will be moving into the non-user category only, whereas movement out will be from non-user category as well as all drug user category. We assume that the rate at which individuals are moving into and out of the population are the same, that is we assume that the population of individuals of age 15–30 does not experience serious fluctuations, thus keeping P constant. We next give a full description of the model.

5. The dynamical system representing NERA

We define our parameters (all real and positive) in Table 1 : For simplicity, we normalize P to 1 unit. Thus, we define a set of variables (*N,E,R,A*) such that $N(t)$ = the proportion of non-user in the population, $E(t)$ = the proportion of experimental user in the population, $R(t)$ = the proportion of recreational user in the population and $A(t)$ = the proportion of addicts in the population, where

$$N(t) + E(t) + R(t) + A(t) = 1. \tag{5.1}$$

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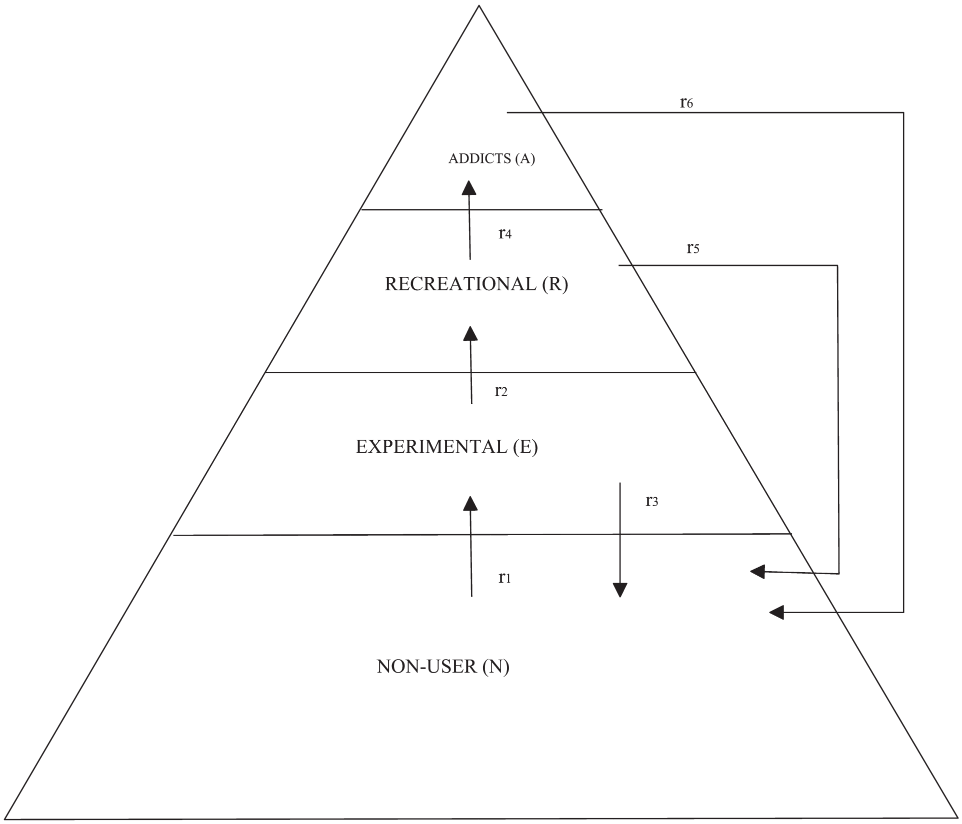

FIG. 1. Schematic representation of drug consumption.

Hence, N, E, R and A must satisfy the following system of differential equations:

$$
\begin{aligned}
\frac{dN}{dt} &= \beta - (\beta + r_1 E + r_1 R)N + r_3 E + r_5 R + r_6 A, \\
\frac{dE}{dt} &= -(\beta + r_3 - r_1 N + r_2 R)E + r_1 N R, \\
\frac{dR}{dt} &= -(\beta + r_4 + r_5 - r_2 E)R, \\
\frac{dA}{dt} &= r_4 R - (\beta + r_6)A. \qquad (5.2)
\end{aligned}
$$

We take $\beta = \frac{1}{15}$ and we calibrate the model by estimating the parameters $r_1, \ldots, r_6$ using the genetic algorithm and the data from Fig. 2 (Korf, 2001) as explained in Korimboccus (2010). The fitness function used is one which minimises the error our model creates. MATLAB's Optimtool is used and the fitness function is inserted in the genetic algorithm. We hence obtain the set of values in Table 2.

Using the parameters given in Table 2, we solve (5.2) for the critical points using Maple. Two critical points are found: (i) $(N^*, E^*, R^*, A^*) = (1, 0, 0, 0)$, the drug-free equilibrium, and

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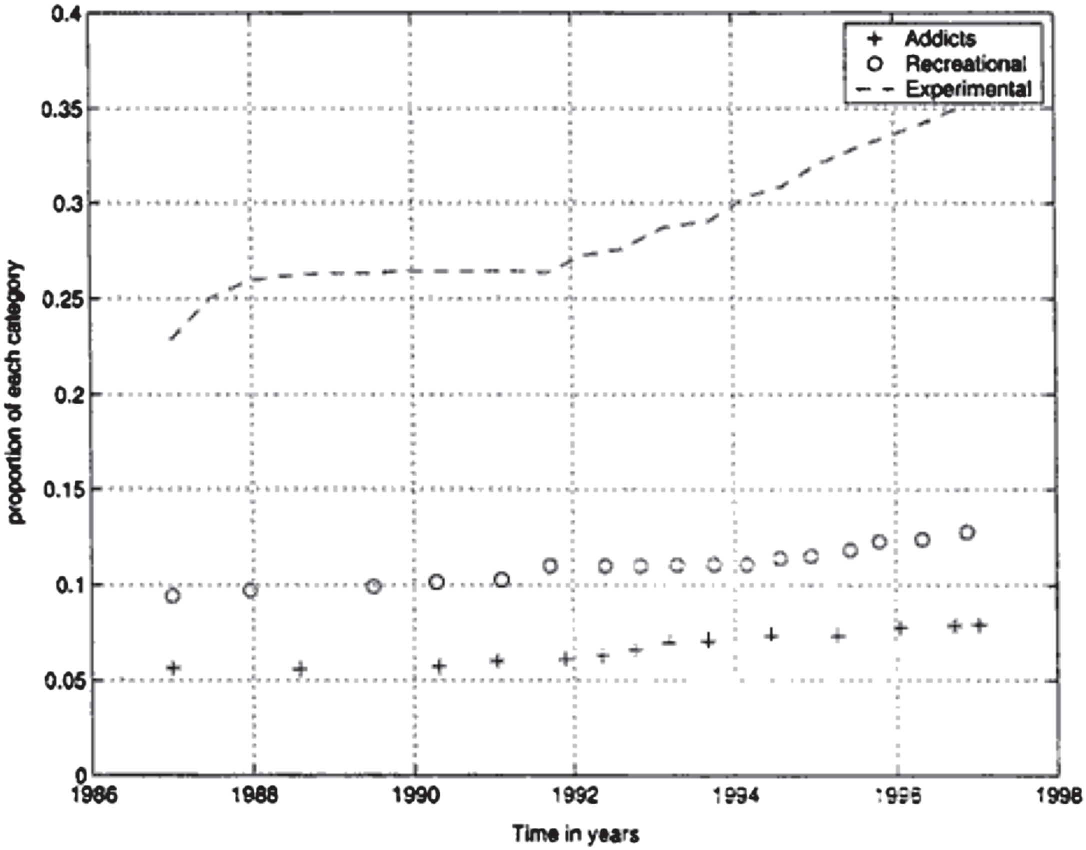

FIG. 2. Trends in Cannabis use for the three different Categories of Drug-Users in the 12+ population in Amsterdam, Korf (2001).

TABLE 1 *Meaning of parameters for the legalization case in the NERA model*

Parameter	Physical meaning
r_1	Influence rate of drug users (E and R) on non-users
r_2	Influence rate of recreational users on experimental users
r_3	Rate at which experimental users quit drugs
r_4	Rate at which recreational users change to addicts
r_5	Rate at which recreational users quit drugs
r_6	Rate at which addicts quit drugs
β	Rate of moving in (or out of) the population due to ageing

(ii) $(N^*, E^*, R^*, A^*) = \left(\frac{\beta+r_3}{r_1}, \frac{r_1-r_3-\beta}{r_1}, 0, 0\right)$, the recreational-addict-free equilibrium because the population consist only of non-users and experimental users. In the sections that follows, we conduct the stability analysis of each of the two critical points.

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TABLE 2 *Parameter values for the legal case*

Parameter	Value
r_1	0.446
r_2	0.5
r_3	0.17
r_4	0.059
r_5	0.002
r_6	0.025
β	$\frac{1}{15}$

6. Analysing the drug-free equilibrium

The Jacobian for the equilibrium point $(1, 0, 0, 0)$ is given by:

$$J(1,\ 0,\ 0,\ 0)=\begin{bmatrix} -\beta & -r_1+r_3 & -r_1+r_5 & r_6 \\ 0 & -(\beta+r_3-r_1) & r_1 & 0 \\ 0 & 0 & -(\beta+r_4+r_5) & 0 \\ 0 & 0 & r_4 & -(\beta+r_6) \end{bmatrix}$$

with eigenvalues: $\lambda_1 = -\beta$, $\lambda_2 = -\beta - r_3 + r_1$, $\lambda_3 = -r_4 - r_5 - \beta$ and $\lambda_4 = -\beta - r_6$.

Clearly, $\lambda_1 < 0$, $\lambda_3 < 0$ and $\lambda_4 < 0$. For the critical point to be asymptotically stable, we require $\lambda_2 < 0$. Thus,

$$r_1 - (\beta + r_3) < 0 \tag{6.1}$$

or

$$\frac{r_1 - r_3}{\beta} < 1. \tag{6.2}$$

The quantity $\gamma = \frac{r_1 - r_3}{\beta}$ can be taken as the average number of individuals from the non-user category, who are influenced by a member of the drug user population to become experimental users. Our study reveals that when

- $\gamma < 1$, the drug-free equilibrium is asymptotically stable as all eigenvalues are negative when the Jacobian is computed and
- when $\gamma > 1$, the drug-free equilibrium is no longer stable as some eigenvalues are negative while others are positive.

When $\gamma = 1$, only the critical point $(1,\ 0,\ 0,\ 0)$ is valid, while two others are invalid because the values of N, E, R and A are then not between 0 and 1. The critical point $(1, 0, 0, 0)$ changes from an asymptotically stable node at $\gamma < 1$ to an unstable saddle point at $\gamma > 1$. This means that when the average number of individuals who are influenced to become experimental users by a member of the drug user population is less than 1, the population approaches a drug-free population asymptotically. However, when γ exceeds 1, then a drug-free population ceases to exist. The crucial issue to address is whether the drug-free equilibrium is possible or not at $\gamma = 1$. We thus proceed to determine the stability of the critical point $(1, 0, 0, 0)$, when the bifurcation occurs at $\gamma = 1$.

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TABLE 3 *Parameter values for* $\gamma = 1$

Parameter	Original value	New value
r_1	0.446	0.36
r_2	0.5	0.5
r_3	0.17	0.30
r_4	0.059	0.059
r_5	0.002	0.002
r_6	0.025	0.025
β	$\frac{1}{15}$	0.06

The following changes are made to the parameters to fit that particular case $\gamma = 1$. The changes are given in Table 3:

The Jacobian matrix $J(1, 0, 0, 0)$ is now given as:

$$J(1,0,0,0)= \begin{bmatrix} -0.06 & -0.06 & -0.358 & 0.025 \\ 0 & 0 & 0.36 & 0 \\ 0 & 0 & -0.121 & 0 \\ 0 & 0 & 0.059 & -0.085 \end{bmatrix}.$$

The following eigenvalues are obtained: $\lambda_1 = -0.06$, $\lambda_2 = 0$, $\lambda_3 = -0.085$ and $\lambda_4 = -0.121$. Because of the presence of 1 zero eigenvalue, no conclusion can be reached pertaining to the stability of the critical point $(1, 0, 0, 0)$ at that value as the linearization test fails. We thus seek for a Lyapunov function for the dynamical system.

We choose the continuous function

$$V(N, E, R, A) = (N-1)^2 + E^2 + R^2 + A^2, \tag{6.3}$$

so that

$$\frac{dV}{dt} = \frac{\partial V}{\partial N}\frac{dN}{dt} + \frac{\partial V}{\partial E}\frac{dE}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt} + \frac{\partial V}{\partial A}\frac{dA}{dt}. \tag{6.4}$$

Equation (6.6) is equivalent to

$$V(N, E, R, A) = (N-1)^2 + E^2 + R^2 + A^2, \tag{6.5}$$

so that

$$\frac{dV}{dt} = \frac{\partial V}{\partial N}\frac{dN}{dt} + \frac{\partial V}{\partial E}\frac{dE}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt} + \frac{\partial V}{\partial A}\frac{dA}{dt}. \tag{6.6}$$

Equation (6.6) is equivalent to

$$\begin{aligned}\frac{dV}{dt} =\, & 2(N-1)(\beta - (\beta + r_1 E + r_1 R)N + r_3 E + r_5 R + r_6 A) + \\ & 2E(-(\beta + r_3 - r_1 N + r_2 R)E + r_1 N R) + \\ & 2R(-(\beta + r_4 + r_5 - r_2 E)R) + 2A(r_4 R - (\beta + r_6)A).\end{aligned} \tag{6.7}$$

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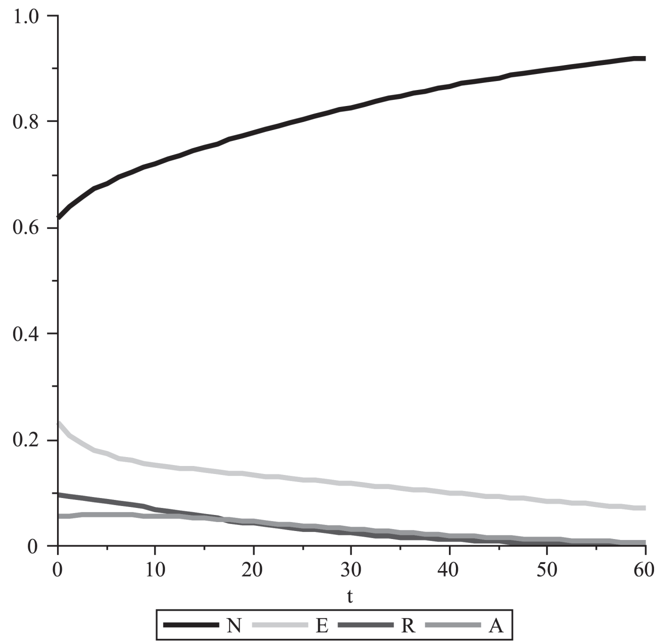

FIG. 3. Evolution of $N(t)$, $E(t)$, $R(t)$ and $A(t)$ at the bifurcation value $\gamma = 1$.

We next determine the sign of the (6.7) by using the optimization tool in MAPLE, to find the maximum of the function $\frac{dV}{dt}$, subject to the following constraints: $0 \leqslant N \leqslant 1$, $0 \leqslant E \leqslant 1$, $0 \leqslant R \leqslant 1$ and $0 \leqslant A \leqslant 1$.

$$\frac{dV}{dt}|_{\max} = -2.09703105023660286 \times 10^{-32}.$$

Since V is a continuous function, we deduce that $\frac{dV}{dt}$ are negative. Therefore, the critical point is still asymptotically stable at the bifurcation value $\gamma = 1$. Hence when the average number of individuals who are influenced by a member of the drug user population to become experimental users is 1, the drug-free equilibrium is still possible after a certain time step.

Figure 3 shows the graph of $N(t)$, $E(t)$, $R(t)$ and $A(t)$ at that bifurcation value $\gamma = 1$.

In the section that follows, we consider the critical point $\left(\frac{\beta+r_3}{r_1}, \frac{r_1-r_3-\beta}{r_1}, 0, 0\right)$, which is the recreational-addict-free equilibrium.

7. Stability analysis of the recreational-addict free equilibrium

The Jacobian at the recreational-addict-free equilibrium is given by:

$$J\left(\frac{\beta+r_3}{r_1}, \frac{r_1-r_3-\beta}{r_1}, 0, 0\right) = \begin{bmatrix} -r_1 + r_3 & -\beta & -\beta - r_3 + r_5 & r_6 \\ r_1 - \beta - r_3 & 0 & \frac{r_2(-r_1+\beta+r_3)}{r_1} + \beta + r_3 & 0 \\ 0 & 0 & -\beta - r_4 - r_5 - \frac{r_2(-r_1+\beta+r_3)}{r_1} & 0 \\ 0 & 0 & r_4 & -\beta - r_6 \end{bmatrix}.$$

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The following eigenvalues are obtained: $\lambda_1 = -\frac{\beta r_1 + r_4 r_1 + r_5 r_1 - r_2 r_1 + r_2 \beta + r_2 r_3}{r_1}$, $\lambda_2 = -\beta - r_6$, $\lambda_3 = -\beta$ and $\lambda_4 = -r_1 + \beta + r_3$.

Clearly, λ_2 and λ_3 are negative. λ_4 is negative because recreational-addict-free equilibrium occurs only when $\gamma > 1$. Hence, since $\lambda_4 = -r_1 + r_3 + \beta$ and $\gamma = \frac{r_1 - r_3}{\beta} > 1$, $\lambda_4 < 0$. λ_1 will be negative provided

$$-\frac{\beta r_1 + r_4 r_1 + r_5 r_1 - r_2 r_1 + r_2 \beta + r_2 r_3}{r_1} < 0$$

or

$$\frac{r_2(r_1 - r_3 - \beta)}{r_1(\beta + r_4 + r_5)} < 1.$$

We let $\mu = \frac{r_2(r_1 - r_3 - \beta)}{r_1(\beta + r_4 + r_5)}$. μ is the proportion of recreational users that exert an influence on non-users via the experimental users, before leaving the category.

Our analysis reveals that drug-free equilibrium exists when $\gamma < 1$. For $\gamma > 1$ and $\mu < 1$, the drug-free equilibrium is unstable, while the recreational-addict-free equilibrium is asymptotically stable. That is, the population is recreational–addict free. For $\gamma > 1$ and $\mu > 1$, both drug-free equilibrium and recreational-addict-free equilibrium are unstable, and the population is made up of all four categories.

From the definition of μ, and the latter inferences, it is clear that the legalization of drugs increases the influence of recreational users on non-users via experimental users. In fact, recreational users will influence experimental users to a significant extent, as compared to the influence they will exert directly on non-users. Thus, $r_2 >> r_1$ and hence μ will increase significantly. Hence, they will exert an influence on non-users via experimental users for a longer time before leaving the recreational category.

In such a society, the campaign of prevention must thus be focused on the experimental users in order to reduce μ.

The different possibilities of γ and μ are analysed.

1. $\gamma > 1$ and $\mu = 1$
2. $\gamma = 1$ and $\mu < 1$
3. $\gamma = 1$ and $\mu = 1$
4. $\gamma = 1$ and $\mu > 1$

However, as long as $\gamma = 1$, μ will always be equal to zero. This can be numerically explained by the fact that $\gamma = \frac{r_1 - r_3}{\beta} = 1$ and therefore $r_1 - r_3 - \beta = 0$ or $\mu = 0$.

Hence, the cases $\gamma = 1$ and $\mu = 1$ and $\gamma = 1$ and $\mu > 1$ are impossible, while the case $\gamma = 1$ and $\mu < 1$ will be possible only if $\mu = 0$. But the latter will be exactly the case $\gamma = 1$ analysed previously.

Only one case remains: $\gamma > 1$ and $\mu = 1$. Physically, this is equivalent to: each member of the drug user population is influencing more than one individual from the non-user population to try drugs and each recreational user is influencing the non-users via the experimental users before leaving the category. Here, drug users exert a maximum influence on the population susceptible to try drugs.

Case: $\gamma > 1$ and $\mu = 1$

The changes brought to the parameters are given in Table 4:

For this case, two critical points are obtained: the drug-free equilibrium $(1, 0, 0, 0)$ and the recreational-addict-free equilibrium (0.4444444444, 0.5555555556,0,0).

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Using the same procedure as before, we see that $(1, 0, 0, 0)$ is unstable, while recreational-addict-free equilibrium has eigenvalues $\lambda_1 = -0.2$, $\lambda_2 = 0.06$, $\lambda_3 = -0.085$ and $\lambda_4 = 0$. Since there is a zero eigenvalue present, we make use of the Lyapunov function (6.5). Since the maximum of the function $\frac{dV}{dt}$ is $-6.71746933518942846 \times 10^{-32}$, from the Lyapunov Stability Theorem, we deduce that recreational-addict-free equilibrium is asymptotically stable.

Figure 4 shows that recreation-addict-free equilibrium is attained after more than 60 time steps. It is clear that the effects of prevalence of drugs in a given society can last for quite long before it eventually disappears. This is confirmed by what happened in China in the early 1900s when millions of the population were addicted to opium. It took China more than 50 years to eradicate opium. This drug is now illegal in that country (How China got rid of opium, 1977).

TABLE 4 *Parameter values for $\gamma > 1$ and $\mu = 1$*

Parameter	Original value	New value
r_1	0.446	0.360
r_2	0.5	0.2484
r_3	0.17	0.1
r_4	0.059	0.07
r_5	0.002	0.008
r_6	0.025	0.025
β	$\frac{1}{15}$	0.06

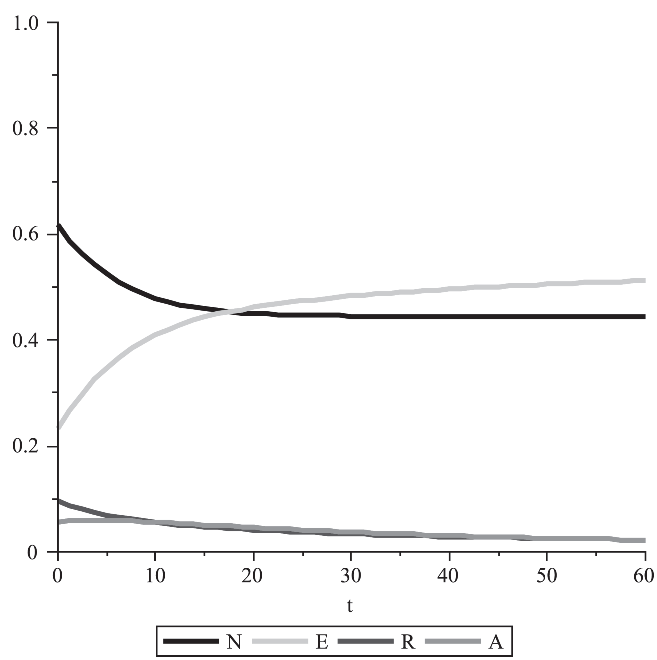

FIG. 4. Evolution of N, E, R and A for the case $\gamma > 1$ and $\mu = 1$.

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Furthermore, we can see that the drug-free equilibrium does not exist at all in such a society since it is an unstable critical point.

Next, we investigate whether we can reach a situation of almost zero drug consumption in a society by declaring drug consumption as illicit.

8. The illegal case

After examining the legal case, we now move on to the illegal case. As compared to the former, the latter takes into account only the influence that experimental users have on non-users. The influence of non-users by the recreational users is regarded as insignificant due to the low interaction between these two categories, on the reason being that recreational users may be perceived as ‘bad people’ by non-users. Furthermore, laws against drug users may act as a deterrent for non-users and experimental users (Korimboccus, 2010).

The illegal case is given by the following system of differential equations:

$$\frac{dN}{dt} = \beta - (\beta + r_1 E)N + r_3 E + r_5 R + r_6 A, \tag{8.1}$$

$$\frac{dE}{dt} = -(\beta + r_2 + r_3 - r_1 N)E, \tag{8.2}$$

$$\frac{dR}{dt} = -(\beta + r_4 + r_5)R + r_2 E, \tag{8.3}$$

$$\frac{dA}{dt} = r_4 R - (\beta + r_6)A, \tag{8.4}$$

where, using the genetic algorithm, the values for $r_1, \ldots, r_6$ are found and given in Table 5.

8.1 *Finding the critical points of the system*

The critical points are found by equating the left hand side of (8.1), (8.2), (8.3) and (8.4) to zero. The following set of equations is obtained:

$$\beta - (\beta + r_1 E)N + r_3 E + r_5 R + r_6 A = 0, \tag{8.5}$$

$$-(\beta + r_2 + r_3 - r_1 N)E = 0, \tag{8.6}$$

TABLE 5 *Physical interpretation of parameters in the illegal case*

Parameter	Physical meaning
r_1	Influence rate of experimental users on non-users
r_2	Rate at which experimental users change to recreational users
r_3	Rate at which experimental users quit drugs
r_4	Rate at which recreational users change to addicts
r_5	Rate at which recreational users quit drugs
r_6	Rate at which addicts quit drugs
β	Rate of moving in (or out of) the population due to ageing

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TABLE 6 *Parameter values for the illegal case*

Parameter	Value
r_1	0.472
r_2	0.119
r_3	0.026
r_4	0.129
r_5	0.081
r_6	0.112
β	$\frac{1}{15}$

$$-(\beta + r_4 + r_5)R + r_2 E = 0, \tag{8.7}$$

$$r_4 R - (\beta + r_6)A = 0. \tag{8.8}$$

Solving yields the critical points (1, 0, 0, 0) and $\left(\frac{\beta+r_2+r_3}{r_1}, -\frac{(\beta^2+\beta r_6+\beta r_4+r_4r_6+r_5\beta+r_5r_6)(r_2+\beta-r_1+r_3)}{r_1(\beta r_2+\beta^2+\beta r_6+r_5r_6+r_4\beta+r_5\beta+r_4r_6+r_2r_6+r_4r_2)},\right.$ $\left.-\frac{r_2(r_2+\beta-r_1+r_3)(\beta+r_6)}{r_1(\beta r_2+\beta^2+\beta r_6+r_5r_6+r_4\beta+r_5\beta+r_4r_6+r_2r_6+r_4r_2)}, -\frac{r_4r_2(r_2+\beta-r_1+r_3)}{r_1(\beta r_2+\beta^2+\beta r_6+r_5r_6+r_4\beta+r_5\beta+r_4r_6+r_2r_6+r_4r_2)}\right)$.

The eigenvalues for the second critical point are quite complicated and we will consider that for future work. In the present work, we analyse the drug-free critical point.

8.2 *Analysing the drug-free equilibrium*

The Jacobian matrix for the system is given by:

$$J = \begin{bmatrix} -\beta - r_1 E & -r_1 N + r_3 & r_5 & r_6 \\ r_1 E & -\beta - r_2 - r_3 + r_1 N & 0 & 0 \\ 0 & r_2 & -\beta - r_4 - r_5 & 0 \\ 0 & 0 & r_4 & -\beta - r_6 \end{bmatrix}.$$

Replacing the values $(N^*, E^*, R^*, A^*) = (1, 0, 0, 0)$, in the Jacobian matrix gives

$$J = \begin{bmatrix} -\beta & -r_1 + r_3 & r_5 & r_6 \\ 0 & -\beta - r_2 - r_3 + r_1 & 0 & 0 \\ 0 & r_2 & -\beta - r_4 - r_5 & 0 \\ 0 & 0 & r_4 & -\beta - r_6 \end{bmatrix}.$$

We next find the eigenvalues of that Jacobian matrix using the characteristic equation

$$(\beta + \lambda)(\beta + r_2 + r_3 - r_1 + \lambda)(\beta + r_4 + r_5 + \lambda)(\beta + r_6 + \lambda) = 0. \tag{8.9}$$

Solving this equation results in the following eigenvalues: $\lambda_1 = -\beta$, $\lambda_2 = -\beta - r_2 - r_3 + r_1$, $\lambda_3 = -\beta - r_4 - r_5$ and $\lambda_4 = -\beta - r_6$.

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TABLE 7 *Change in parameter values for the case $\xi = 1$*

Parameter	Original value	New value
r_1	0.472	0.372
r_2	0.119	0.27
r_3	0.026	0.042
r_4	0.129	0.129
r_5	0.081	0.081
r_6	0.112	0.112
β	$\frac{1}{15}$	0.06

Clearly, λ_1, λ_3 and λ_4 are negative, while λ_2 is negative provided

$$-\beta - r_2 - r_3 + r_1 < 0. \tag{8.10}$$

This is equivalent to $\xi < 1$, where $\xi = \frac{r_1 - r_2 - r_3}{\beta}$.

In fact, ξ defines the average number of individuals in the non-user category who are being influenced by a member of the experimental category to try drugs. For drug-free equilibrium to be asymptotically stable, inequality (8.10) must be satisfied. From Korimboccus (2010), at $\xi = 1$, the system bifurcates but no indication is given as to whether the critical point is still stable at that specific point. Then when $\xi > 1$, the critical point becomes unstable and a drug-free population is no longer possible.

Consequently, we examine the case $\xi = 1$ and investigate the stability at that point.

Case: $\xi = 1$

The following changes are made to the parameter values to fit that particular case.

Only the critical point (1, 0, 0, 0) exists and the Jacobian matrix is given by:

$$J(1,0,0,0) = \begin{bmatrix} -0.06 & -0.330 & 0.089 & 0.112 \\ 0 & 0 & 0 & 0 \\ 0 & 0.27 & -0.278 & 0 \\ 0 & 0 & 0.129 & -0.172 \end{bmatrix}.$$

The following eigenvalues are obtained $\lambda_1 = -0.06$, $\lambda_2 = -0.172$, $\lambda_3 = -0.278$ and $\lambda_4 = 0$.

At that point, however, since linearization fails, as was previously the case with $\gamma = 1$, we use the Lyapunov Stability Theorem to determine its stability.

Using the Lyapunov function (6.5) and its derivative (6.7), and the Optimization tool in MAPLE, we obtain $\frac{dV}{dt} = -2.02801422294992067 \times 10^{-31}$, implying that the critical point is still asymptotically stable at that value $\xi = 1$.

Figure 5 shows the evolution of $N(t)$, $E(t)$, $R(t)$ and $A(t)$ at that point $\xi = 1$. In fact, the results confirm that it is possible to reach a drug-free equilibrium in the case when drugs are illegal.

9. Concluding remarks

Finally, we present the results of the numerical experiment in which we solve (5.2) with parameters given by Table 2 using MAPLE.

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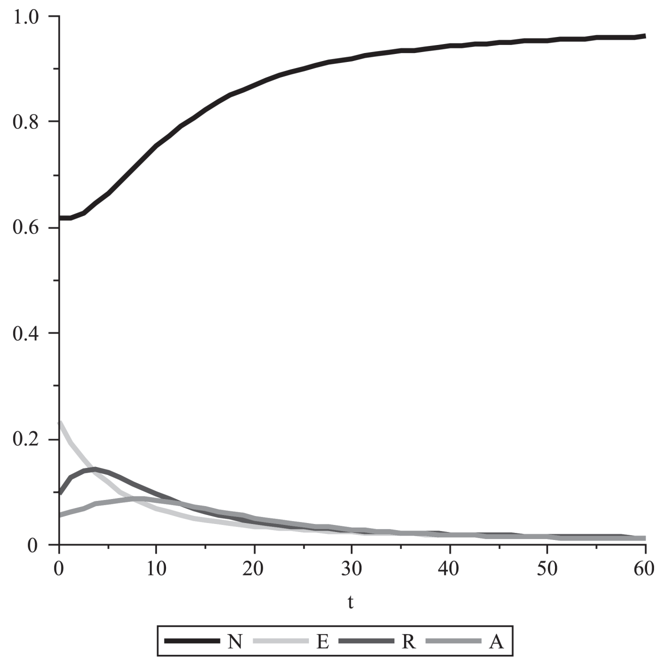

FIG. 5. Evolution of $N(t)$, $E(t)$, $R(t)$ and $A(t)$ at the bifurcation value $\xi = 1$.

Figure 6 shows our results (solid lines) superimposed on the data from Korf (2001) (dotted) for the first 12 timesteps (1986–1998). The results agree quite accurately with the observed data confirming that (5.2) can be a precious tool in forecasting of illicit drug consumption in a given population. Also, since the parameters of Table 2 can be adjusted and corresponding results of (5.2) obtained, the NERA model justifies its use as a Policy Control Mechanism in tackling illicit drug consumption in any population.

In this paper, we described and analyzed a nonlinear dynamical system which can be used to assess the impact of the legalization of consumption of certain illicit drugs in a given population.

As future work, we propose to redefine the model to a stochastic one. The model can be made stochastic. In fact, we assume that a random number of individuals migrate from the non-user category to the experimental one, then we can represent the system given by (5.2) as

$$\begin{aligned} dE &= [r_1 N - (r_2 + r_3 + r_4)\, E]\, dt + r_1 N \eta dW_t, \\ dR &= [r_2 E - (r_5 + r_6) R]\, dt, \\ dA &= [r_3 E + r_5 R - r_7 A]\, dt, \end{aligned} \tag{9.1}$$

where W_t is the Wienerprocess Evans and η the noise coefficient Evans that determines the size of the stochastic term $r_1 N \eta \frac{dW_t}{dt}$. With the new term, the system of (5.2) will be transformed to a system of stochastic differential equations. In a future work, we propose to analyse the latter system.

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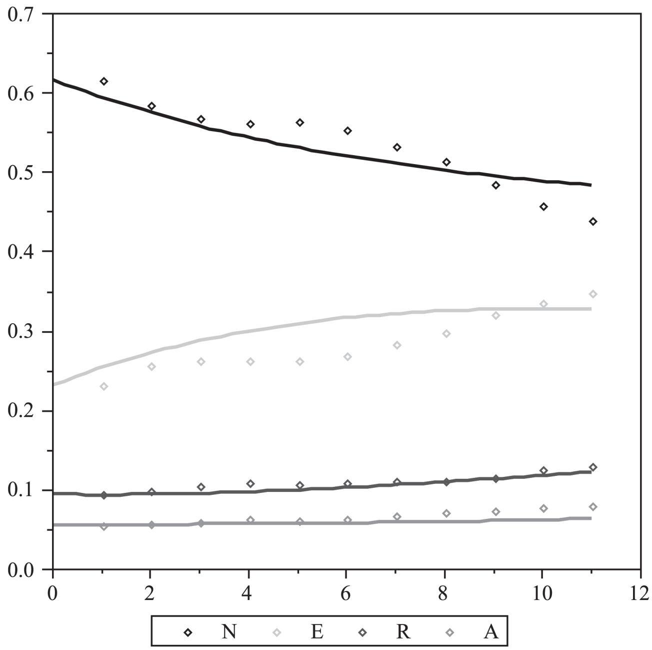

FIG. 6. Graphs obtained using the linear model (solid lines) superimposed on the graphs showing the real-life trends (dotted lines).

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